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Scalar $\lambda\phi^4$ Model with Nontrivial Topology

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Effects of nontrivial topology $S^1 \times R^3$ of space-time for a scalar self-interacting $\lambda\phi^4$ model in the perturbative regime are discussed. Asymptotic properties of the coupling constant for untwisted and twisted field configurations are investigated.

1. INTRODUCTION

Quantum field models are sensitive to both the local and the global space-time structures. The local structure of space-time is connected with curvature, and the global one with topology. The interest in models with nontrivial geometries and topologies is based on the attempts to describe the interaction of quantum fields with gravity [1–3], and (or) in the presence of surfaces [4–8]. The role of topological structure of the space-time manifold was considered in a number of articles: interacting quantum fields in the perturbative regime were studied in refs. 9–13 with regard to the symmetry breaking and mass generation in self-interacting and gauge models, the problem of vacuum polarization and causality in electrodynamics [14], and in models with dynamical symmetry breaking in the nonperturbative regime [16, 17].

The renormalization of interacting quantum fields with periodic (antiperiodic) identification in one of the spatial coordinates [12, 15] in the perturbative regime includes a topological parameter, and the topology influences the behavior of physical parameters of the models. The construction of thermo field models is also connected with changing the initial space-time structure of field models through the introduction of the temperature by compactification of the time coordinate, which allows one not only to construct the thermodynamics of quantum systems [18], but also to study the variations

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of physical parameters of these models with temperature [19–21], finite-temperature mass corrections for the electron in QED [22], plasmon mass in the relativistic fermion gas [23], symmetry restoration and phase transitions [26, 27, 29], etc. From the technical point of view thermal field models and field models with nontrivial topology of space-time with one parameter can be treated in the same manner. The nontrivial topology as well as temperature corrections affect the Green's functions of the fields without any radical changing of the Feynman rules, which allows us to use the methods of standard quantum field theory [28].

The purpose of this article is to study scalar-field model with $\lambda\phi^4$ interaction in space-time with nontrivial topology in perturbation theory; to consider the problem of renormalization of the model in space-time with $S^1 \times R^3$ topology; to study the behavior of the physical parameters of the model with respect to scaling the topological parameters; and to consider the finite-temperature regime for this model.

The study of this model from the aspects mentioned above might offer new insight into the structure renormalizations and dependence of the physical parameters of quantum field models on the topology in perturbation theory. The method used for renormalization of this model is based on the loop computations with the Feynman propagator similar to its "real time" form of thermo field dynamics (TFD) [22, 26, 30] and minimal subtraction renormalization procedure [32, 24]. Such a formulation of the Feynman propagator considerably simplifies computations of Feynman graphs and the procedure of renormalization, making them similar to relatively straightforward methods of TFD. In order to study the behavior of the coupling constant of the model we use the modified renormalization group (RG) technique which affects the topoligical structure of the space-time.

The article is organized in the following way. Two-loop renormalizations, the structure of counterterms, and the behavior of physical parameters of the self-interacting $\lambda\phi^4$ model in $S^1 \times R^3$ space-time topology with untwisted and twisted field configurations are considered in Section 2. Asymptotic properties of the model are analyzed in Section 3. The connection with finite-tempereature field theory and the properties of the model are discussed in Section 4.

2. RENORMALIZATIONS OF $\lambda\phi^4$ MODEL WITH NONTRIVIAL TOPOLOGY

In this section, we consider the methods of two-loop renomalization of the scalar self-interacting $\lambda\varphi^4$ model with $S^1 \times R^3$ topology. Let the Euclidean unrenormalized Lagrangian be written as

$$L = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_B^2\phi^2 - \frac{\lambda_B}{4!}\phi^4 \tag{1}$$

where m_B and λ_B are bare mass and coupling constants.

To develop the perturbation formalism for the model we assume

$$m_B^2 = \delta m^2 + m_R^2$$

$$\lambda_B = \delta\lambda + \lambda_R \tag{2}$$

where m_R and λ_R are the renormalized mass parameter and the coupling constant, and δm^2 and $\delta\lambda$ are mass and coupling constant counterterms. Then one can write (1) as the sum of unperturbed and interacting parts

$$L = L_0 + L_{\text{int}} \tag{3}$$

where

$$L_0 = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_R^2\phi^2 \tag{4}$$

$$L_{\text{int}} = -\frac{\delta m^2}{2}\phi^2 - \frac{\lambda_B}{4!}\phi^4 \tag{5}$$

Let us introduce the topology $S^1 \times R^3$ in flat space-time by making a compactification with respect to one spatial coordinate $x^1 \in [0, \zeta]$. This topology allows us to introduce untwisted and twisted fields as $\phi(x^1 = 0, \mathbf{x}) = \pm\phi(x^1 = \zeta, \mathbf{x})$.

The generating functional of the model will have the following form:

$$Z_E[\zeta] = \int_{\phi=\pm\phi} D\phi \exp\left[\int_0^\zeta dx^1 \int_{R^3} d^3\mathbf{x}\,(L(\phi, \partial\phi) + \phi(x)J(x))\right] \tag{6}$$

The quadratic part of this action is

$$S_0 = -\frac{1}{2}\int_0^\zeta dx^1 \int_{R^3} d^3\mathbf{x}\,[(\partial\phi)^2 + m_R^2\phi^2] \tag{7}$$

To get the Feynman propagator from S_0 we expand $\phi(x^1, \mathbf{x})$ in a Fourier series

$$\phi(x^1, \mathbf{x}) = \frac{1}{\zeta}\sum_{n=-\infty}^{\infty}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\exp[i\omega_n x^1 + i\mathbf{k}\mathbf{x}]\phi_n(\mathbf{k}) \tag{8}$$

where

$$\omega_n = \begin{cases} (2\pi/\zeta)(n + 1/2) \\ (2\pi/\zeta)n \end{cases} \tag{9}$$

defines periodic and antiperiodic field configurations.

With the help of the equation

$$\frac{1}{\zeta}\int_0^{\zeta} dx^1 \exp[i(\omega_n - \omega_{n'})x^1] = \delta_{nn'} \tag{10}$$

we write the quadratic part of the action

$$S_0 = -\frac{1}{2}\sum_{n=-\infty}^{\infty}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\phi_n(\mathbf{k})[\omega_n^2 + \mathbf{k}^2 + m_B^2]\phi_{-n}(-\mathbf{k}) \tag{11}$$

or in the form of a scalar product on function space as

$$S_0 = -\frac{1}{2}(\phi, D\phi) \tag{12}$$

where

$$D = \omega_n^2 + \mathbf{k}^2 + m_R^2 \tag{13}$$

The Feynman propagator Δ in the momentum space is connected with D by the inverse transform; therefore

$$\Delta(\omega_n, \mathbf{k}) = \frac{1}{\omega_n^2 + \mathbf{k}^2 + m_R^2} \tag{14}$$

and in the position space we get

$$\Delta_F(x - x') = \frac{1}{\zeta}\sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{\exp[i\omega_n(x^1 - x'^1) + i\mathbf{k}(\mathbf{x} - \mathbf{x}')]}{\omega_n^2 + \mathbf{k}^2 + m_R^2} \tag{15}$$

This expression may be written in as

$$\Delta_F(x) = \int_{\zeta} \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2 + m_R^2}\exp[ikx] \tag{16}$$

where the momentum is $k_\mu = (\omega_n, \mathbf{k})$ and the symbol of integration is

$$\int_{\zeta}\frac{d^4k}{(2\pi)^4} = \frac{1}{\zeta}\sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3} \tag{17}$$

Adding here the vertex coupling constant in the form $(-\lambda_R)$ and the expression for the δ-function

$$(2\pi)^4\delta^4(k^1, k^2, \ldots) = (2\pi)^3\zeta\delta_{\omega_{n1},\omega_{n2},\ldots}\,\delta^3(\mathbf{k}^1, \mathbf{k}^2, \ldots) \tag{18}$$

we completely define the modifications for Feynman rules at $S^1 \times R^3$ space-time.

An alternative way to study this model in the perturbative regime is based on the modification of the Feynman propagator to the four-dimensional momentum form

$$\Delta(x, \zeta) = \int \frac{d^4k}{(2\pi)^4}\, \Delta(k, \zeta)\, \exp(ikx) \tag{19}$$

with

$$\Delta(k, \zeta) = \frac{1}{k^2 + m_R^2} \pm \frac{2\pi i\delta(k^2 + m_R^2)}{\exp(\zeta|k^1|) \mp 1} = \Delta_0(k) \pm \Delta_\pm(k, \zeta) \tag{20}$$

where $\Delta_0(k)$ is the standard form of Feynman propagator, and $\Delta_\pm(k, \zeta)$ are topologically dependent contributions. This form of tree-level propagator does not include the summation and is convenient for perturbative computations.

To make renormalizations of the model we will use the propagator in the form (20) and vertex $(-\lambda_R\mu^{4-n})$, where the mass parameter μ makes the coupling constant dimensionless for $n \neq 4$. Two and four 1 P.I. point vertices are shown in Figs. 1 and 2.

The contribution of Fig. 1b of the order λ_R to the self-energy is

$$\begin{aligned}\Gamma^{(b)}_{2)}(\zeta) &= \frac{1}{2}\,(-\lambda_R\mu^{4-n}) \int \frac{d^nk}{(2\pi)^n}\, \Delta(k, \zeta)\\ &= -\frac{\lambda_R}{2}\,\mu^{4-n} \int \frac{d^nk}{(2\pi)^n}\, \Delta_0(k) \mp \frac{\lambda_R}{2}\,\mu^{4-n} \int \frac{d^nk}{(2\pi)^n}\, \Delta_\pm(k, \zeta)\end{aligned} \tag{21}$$

We write the first integral of (21) with the help of the equation

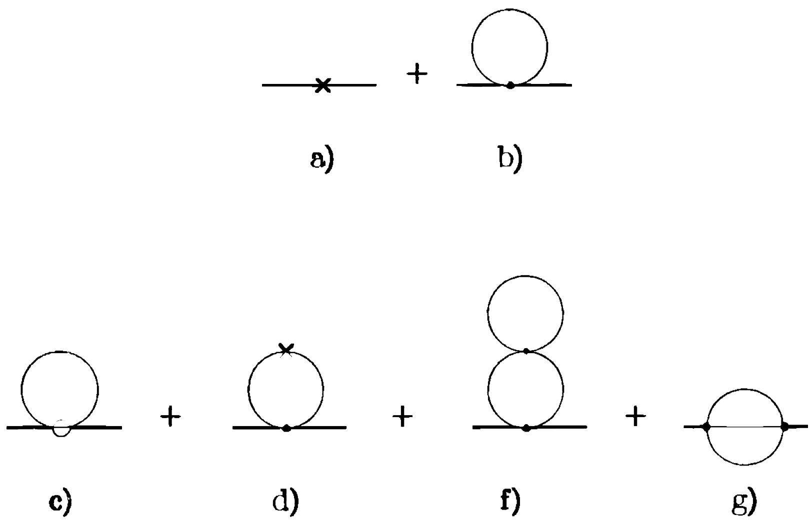

Fig. 1. Diagrams contributing to the self-energy.

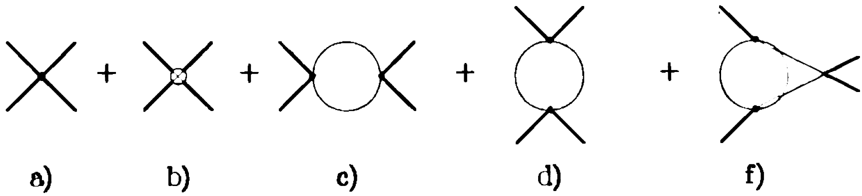

Fig. 2. Vertex diagrams.

$$\int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 + M^2 + 2kp)^d} = \frac{1}{(4\pi)^{n/2}} \frac{\Gamma(d - n/2)}{\Gamma(d)} \frac{1}{(M^2 - p^2)^{d-n/2}} \tag{22}$$

in the form

$$\begin{aligned} &\frac{1}{2}(-\lambda_R \mu^{4-n}) \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 + m_R^2} \\ &= \frac{1}{2}(-\lambda_R \mu^{4-n}) \frac{1}{(4\pi)^{n/2}} \frac{\Gamma(1 - n/2)}{\Gamma(1)} \frac{1}{(m_R^2)^{1-n/2}} \\ &= -\frac{\lambda_R}{2}\left(\frac{m_R^2}{16\pi^2}\right)\left(\frac{m_R^2}{4\pi\mu^2}\right)^{n/2-2} \Gamma\left(1 - \frac{n}{2}\right) \end{aligned} \tag{23}$$

Using the expansions

$$\Gamma\left(1 - \frac{n}{2}\right) = \frac{2}{n-4} + \gamma - 1 \tag{24}$$

and

$$\left(\frac{m_R^2}{4\pi\mu^2}\right)^{n/2-2} = 1 + \left(\frac{n}{2} - 2\right) \ln \frac{m_R^2}{4\pi\mu^2} + O\left(\left(\frac{n}{2} - 2\right)^2\right) \tag{25}$$

we find the resulting equation for Fig. 1b as

$$\begin{aligned} \Gamma_{2)}^{(b)}(\zeta) &= -\lambda_R \frac{m_R^2}{32\pi^2} \left[\frac{2}{n-4} + \gamma - 1 + \ln \frac{m_R^2}{4\pi\mu^2}\right] \\ &\quad \mp \frac{\lambda_R}{2} \mu^{4-n} \Sigma_{\pm}(\zeta) \end{aligned} \tag{26}$$

where

$$\Sigma_{\pm}(\zeta) = \int \frac{d^n k}{(2\pi)^n} \Delta_{\pm}(k, \zeta) \tag{27}$$

is a function of the parameter ζ.

Now we compute contributions of the second order of perturbation. The diagram of Fig. 1c is

$$\Gamma^{(c)}_{2)}(\zeta) = \frac{1}{2}(-\delta\lambda\mu^{4-n}) \int \frac{d^nk}{(2\pi)^n} \Delta(k, \zeta)$$
$$= -\frac{\delta\lambda}{2}\mu^{4-n} \int \frac{d^nk}{(2\pi)^n} \Delta_0(k) \mp \frac{\delta\lambda}{2}\mu^{4-n} \int \frac{d^nk}{(2\pi)^n} \Delta_\pm(k, \zeta) \quad (28)$$

The first integral is found the same way as before. It gives

$$\Gamma^{(c)}_{2)}(\zeta) = -\delta\lambda \frac{m_R^2}{32\pi^2} \left[\frac{2}{n-4} + \gamma - 1 + \ln\frac{m_R^2}{4\pi\mu^2}\right]$$
$$\mp \frac{\delta\lambda}{2}\mu^{4-n}\Sigma_\pm(\zeta) \quad (29)$$

The contribution of Fig. 1d is

$$\Gamma^{(d)}_{2)}(\zeta) = \frac{1}{2}(\lambda_R\mu^{4-n})\delta m^2 \int \frac{d^nk}{(2\pi)^n} \Delta^2(k, \zeta)$$
$$= \frac{1}{2}(\lambda_R\mu^{4-n})\delta m^2 \int \frac{d^nk}{(2\pi)^n} [\Delta_0^2(k) \pm 2\Delta_0(k)\Delta_\pm(k, \zeta) + \Delta_\pm^2(k, \zeta)] \quad (30)$$

The first integral is

$$\frac{1}{2}(\lambda_R\mu^{4-n})\delta m^2 \int \frac{d^nk}{(2\pi)^n} \frac{1}{(k^2 + m_R^2)^2}$$
$$= \frac{1}{2}(\lambda_R\mu^{4-n}) \frac{\delta m^2}{(4\pi)^{n/2}} \frac{\Gamma(2 - n/2)}{\Gamma(2)} \frac{1}{(m_R^2)^{2-n/2}}$$
$$= \frac{\lambda_R}{32\pi^2}\delta m^2 \left(\frac{m_R^2}{(4\pi\mu^2)}\right)^{n/2-2} \Gamma\left(2 - \frac{n}{2}\right) \quad (31)$$

Using the expression

$$\Gamma\left(2 - \frac{n}{2}\right) = -\frac{2}{n-4} - \gamma \quad (32)$$

and expansion (25), we find

$$\frac{1}{2}(\lambda_R\mu^{4-n})\delta m^2 \int \frac{d^nk}{(2\pi)^n} \frac{1}{(k^2+m_R^2)^2}$$
$$= \frac{\lambda_R}{32\pi^2}\delta m^2 \left[-\frac{2}{n-4} - \gamma\right]\left[1 + \frac{n-4}{2}\ln\frac{m_R^2}{4\pi\mu^2}\right] \tag{33}$$

As a result we get the expression for the divergent part of Fig. 1d in the form

$$\Gamma_{2)}^{(d)}(\zeta) = -\frac{\lambda_R}{32\pi^2}\delta m^2\left[\frac{2}{n-4} + \gamma + \ln\frac{m_R^2}{4\pi\mu^2}\right]$$
$$\pm \lambda_R(\delta m^2)\mu^{4-n}\Omega_\pm(\zeta) \tag{34}$$

where

$$\Omega_\pm(\zeta) = \int \frac{d^nk}{(2\pi)^n}\Delta_0(k)\Delta_\pm(k,\zeta) \tag{35}$$

is a finite function.

The two-loop contribution Fig. 1f is

$$\Gamma_{2)}^{(f)}(\zeta) = \frac{1}{4}(-\lambda_R\mu^{4-n})^2\int \frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\Delta^2(k,\zeta)\Delta(q,\zeta) \tag{36}$$

The divergent terms are found from

$$\Gamma_{2)}^{(f)}(\zeta) = \frac{1}{4}(\lambda_R\mu^{4-n})^2\int \frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\Delta_0^2(k)\Delta_0(q)$$
$$\pm \frac{1}{2}(\lambda_R\mu^{4-n})^2\int \frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\Delta_0(k)\Delta_0(q)\Delta_\pm(k,\zeta)$$
$$\pm \frac{1}{4}(\lambda_R\mu^{4-n})^2\int \frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\Delta_0^2(k)\Delta_\pm(q,\zeta) \tag{37}$$

Rewrite (37) as

$$\Gamma_{2)}^{(f)}(\zeta) = \frac{1}{4}(-\lambda_R\mu^{4-n})^2\int \frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\Delta_0^2(k)\Delta_0(q)$$
$$\pm \frac{1}{2}\left[(\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\Delta_0(k)\right]\Omega_\pm(\zeta)$$
$$\pm \frac{1}{4}\left[(\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\Delta_0^2(k)\right]\Sigma_\pm(\zeta) \tag{38}$$

The first integral of the above expression is

$$
\begin{aligned}
&\frac{1}{4}(-\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\Delta_0^2(k)\Delta_0(q)\\
&\quad=\frac{\lambda_R^2}{4}\left[\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\Delta_0^2(k)\right]\left[\mu^{4-n}\int\frac{d^nq}{(2\pi)^n}\Delta_0(q)\right]\\
&\quad=\frac{\lambda_R^2}{4}\left[\frac{1}{16\pi^2}\left(\frac{m_R^2}{4\pi\mu^2}\right)^{n/2-2}\Gamma\left(2-\frac{n}{2}\right)\right]\left[\frac{m_R^2}{16\pi^2}\left(\frac{m_R^2}{4\pi\mu^2}\right)^{n/2-2}\Gamma(1-n/2)\right]\\
&\quad=\frac{\lambda_R^2}{4}\left(\frac{1}{16\pi^2}\right)^2\left(\frac{m_R^2}{4\pi\mu^2}\right)^{n-4}m_R^2\Gamma\left(2-\frac{n}{2}\right)\Gamma\left(1-\frac{n}{2}\right)\\
&\quad=m_R^2\frac{\lambda_R^2}{4}\left(\frac{1}{16\pi^2}\right)^2\left[-\frac{2}{n-4}-\gamma\right]\left[\frac{2}{n-4}-1+\gamma\right]\left[1+(n-4)\ln\frac{m_R^2}{4\pi\mu^2}\right]\\
&\qquad-m_R^2\frac{\lambda_R^2}{4}\left(\frac{1}{16\pi^2}\right)^2\left[\frac{4}{(n-4)^2}+\frac{2}{(n-4)}\left(2\gamma-1+2\ln\frac{m_R^2}{4\pi\mu^2}\right)+\cdots\right]
\end{aligned}\tag{39}
$$

The next two integrals are computed in the same way and the result for divergent contributions in (36) is

$$
\begin{aligned}
\Gamma_{2)}^{(f)}(\zeta)&=-\frac{\lambda_R^2}{(16\pi^2)^2}\frac{m_R^2}{(n-4)^2}\\
&\quad-\frac{\lambda_R^2}{2(16\pi^2)^2}\frac{m_R^2}{(n-4)}\left[2\gamma-1+2\ln\frac{m_R^2}{4\pi\mu^2}\right]\\
&\quad\mp\frac{\lambda_R^2}{32\pi^2}\frac{1}{(n-4)}\mu^{4-n}\Sigma_\pm(\zeta)\pm\frac{\lambda_R^2m_R^2}{16\pi^2}\frac{1}{(n-4)}\mu^{4-n}\Omega_\pm(\zeta)
\end{aligned}\tag{40}
$$

The contribution of Fig. 1g is written as

$$
\Gamma_{2)}^{(g)}(p,\zeta)=\frac{1}{6}(-\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\Delta(k,\zeta)\Delta(q,\zeta)\Delta(k+q+p,\zeta)\tag{41}
$$

Divergent terms in this integral are written as

$$
\begin{aligned}
\Gamma_{2)}^{(g)}(p,\zeta)&=\frac{1}{6}(-\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\Delta_0(k)\Delta_0(q)\Delta_0(k+q+p)\\
&\quad\pm\frac{1}{6}(-\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\Delta_\pm(k,\zeta)\Delta_0(k+q+p)\Delta_0(q)\\
&\quad\pm\frac{1}{6}(-\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\Delta_\pm(q,\zeta)\Delta_0(k)\Delta_0(k+q+p)
\end{aligned}
$$

$$\pm\frac{1}{6}(-\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\Delta_0(k)\Delta_0(q)\Delta_\pm(k+q+p,\zeta) \quad (42)$$

The first integral of the above expression does not include the topological contributions. Its divergent part is written as

$$\begin{aligned}
&\frac{1}{6}(-\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\Delta_0(k)\Delta_0q\Delta_0(k+q+p,\zeta)\\
&= -\frac{\lambda_R^2}{(16\pi^2)^2}\left(\frac{m_R^2}{4\pi\mu^2}\right)^{n-4}\frac{m_R^2}{(n-4)^2}+\frac{\lambda_R^2}{(16\pi^2)^2}\frac{1}{(n-4)}\\
&\quad\times\left[\frac{p^2}{12}+\frac{m_R^2}{2}-(\gamma-1)m_R^2\right]+\cdots\\
&= -\frac{\lambda_R^2}{(16\pi^2)^2}\frac{m_R^2}{(n-4)^2}+\frac{\lambda_R^2}{12(16\pi^2)^2}\frac{1}{(n-4)}(p^2+6m_R^2)\\
&\quad-\frac{\lambda_R^2}{(16\pi^2)^2}\frac{m_R^2}{(n-4)}\left[\gamma-1+\ln\frac{m_R^2}{4\pi\mu^2}\right]+\cdots
\end{aligned} \quad (43)$$

The sum of the contributions with topological parameter is

$$\begin{aligned}
&\frac{1}{6}(-\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\Delta_\pm(k,\zeta)\Delta_0(k+q+p)\Delta_0(q)\\
&\quad+\frac{1}{6}(-\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\Delta_\pm(q,\zeta)\Delta_0(k)\Delta_0k+q+p)\\
&\quad+\frac{1}{6}(-\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\frac{d^nq}{(2\pi)^n}\Delta_0(k)\Delta_0(q)\Delta_\pm(k+q+p,\zeta)\\
&= -\frac{\lambda_R^2\mu^{4-n}}{32\pi^2}\int\frac{d^nk}{(2\pi)^n}\Delta_\pm(k,\zeta)\\
&\quad\times\left[\frac{2}{(n-4)}+\gamma+\frac{2}{3}\int_0^1 dx\ln\frac{m_R^2+(p+k)^2x(1-x)}{4\pi\mu^2}\right.\\
&\quad\left.+\frac{1}{3}\int_0^1 dx\ln\frac{m_R^2+(p-k)^2x(1-x)}{4\pi\mu^2}\right]
\end{aligned} \quad (44)$$

Combining the divergent terms of Fig. 1g, we get

$$\Gamma^{(g)}_{2)}(p, \zeta) = -\frac{\lambda_R^2}{(16\pi^2)^2}\frac{m_R^2}{(n-4)^2} + \frac{\lambda_R^2}{(16\pi^2)^2}\frac{1}{12(n-4)}(p^2 + 6m_R^2)$$
$$-\frac{\lambda_R^2}{(16\pi^2)^2}\frac{m_R^2}{(n-4)}\left[\gamma - 1 + \ln\frac{m_R^2}{4\pi\mu^2}\right]$$
$$\mp\frac{\lambda_R^2\mu^{4-n}}{16\pi^2}\frac{1}{(n-4)}\Sigma_\pm(\zeta) \tag{45}$$

The contribution of Fig. 2c to the four-point vertex is

$$\Gamma^{(c)}_{4)}(p, \zeta) = \frac{1}{2}(-\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\Delta(k)\Delta(k+p)$$
$$= \frac{1}{2}(-\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\Delta_0(k)\Delta_0(k+p)$$
$$\pm\frac{1}{2}(-\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\Delta_\pm(k)\Delta_0(k+p)$$
$$\pm\frac{1}{2}(-\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\Delta_0(k)\Delta_\pm(k+p)$$
$$+\frac{1}{2}(-\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\Delta_\pm(k)\Delta_\pm(k+p) \tag{46}$$

Rewrite the first integral with the help of the equation

$$\frac{1}{\alpha\beta} = \int_0^1 dx\,\frac{1}{[\alpha x + \beta(1-x)]^2} \tag{47}$$

as

$$\frac{1}{2}(-\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\Delta_0(k)\Delta_0(k+p)$$
$$= \frac{1}{2}(-\lambda_R\mu^{4-n})^2\int_0^1 dx\int\frac{d^nk}{(2\pi)^n}\frac{1}{[(k^2 + 2pkx + (m_R^2 + p^2x)]^2} \tag{48}$$

Using (22), (25), and (32), we find

$$\frac{1}{2}(-\lambda_R\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\Delta_0(k)\Delta_0(k+p)$$
$$= \frac{\lambda_R^2}{32\pi^2}\mu^{4-n}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx\left[\frac{m_R^2 + p^2x(1-x)}{4\pi\mu^2}\right]^{n/2-2}$$

$$= \frac{\lambda_R^2}{32\pi^2}\mu^{4-n}\int_0^1 dx\left[-\frac{2}{n-4}-\gamma\right]\left[1+\frac{n-4}{2}\ln\frac{m_R^2+p^2x(1-x)}{4\pi\mu^2}\right] \tag{49}$$

Then

$$\frac{1}{2}(-\lambda_R^2\mu^{4-n})^2\int\frac{d^nk}{(2\pi)^n}\Delta_0(k)\Delta_0(k+p)$$
$$= -\frac{\lambda_R^2}{16\pi^2}\mu^{4-n}\frac{1}{(n-4)}-\frac{\lambda_R^2}{32\pi^2}\mu^{4-n}\left[\gamma+\int_0^1 dx\ln\frac{m_R^2+p^2x(1-x)}{4\pi\mu^2}\right] \tag{50}$$

As a result we find

$$\begin{aligned}\Gamma^{(c)}_{4)}(p,\zeta) = &-\frac{\lambda_R^2}{16\pi^2}\mu^{4-n}\frac{1}{(n-4)}\\ &-\frac{\lambda_R^2}{32\pi^2}\mu^{4-n}\left[\gamma+\int_0^1 dx\ln\frac{m_R^2+p^2x(1-x)}{4\pi\mu^2}\right]\\ &\pm(\lambda_R\mu^{4-n})^2\Omega_\pm(p)\end{aligned} \tag{51}$$

where

$$\Omega_\pm(p,\zeta) = \int\frac{d^3k}{(2\pi)^3}\Delta_0(k+p)\Delta_\pm(k,\zeta) \tag{52}$$

The momentum p in Fig. 2c is $p = p_1 + p_2$. The vertices in Fig. 2d and 2f have the same structure with momenta $p = p_1 + p_3$ and $p = p_1 + p_4$.

The divergent part of each of these vertices is equal to

$$\Gamma^{\rm div}_{4)} = -\frac{\lambda_R^2\mu^{4-n}}{16\pi^2(n-4)} \tag{53}$$

For the renormalization of the model we assume that $\delta\lambda$, δm^2, and the field renormalization constant Z are written as series [24]:

$$\delta\lambda = \sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\frac{a_{ij}}{(n-4)^i}\lambda_R{}^j \tag{54}$$

$$\delta m^2 = m_R^2\sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\frac{b_{ij}}{(n-4)^i}\lambda_R{}^j \tag{55}$$

and

$$Z = 1 + \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \frac{c_{ij}}{(n-4)^i} \lambda_R{}^j \tag{56}$$

The mass and vertex counterterms and renormalization constant Z (the coefficients a_{ij}, b_{ij}, c_{ij}) are chosen from

$$Z[p^2 + m_B^2 - \Sigma] = \text{finite} \tag{57}$$

and

$$Z^2 \left[\sum_{4-\text{vertex}} \Gamma_4 + O(\lambda_R^3) \right] = \text{finite} \tag{58}$$

in the limit $n \to 4$.

The mass counterterm to the order λ_R is found from (57) with the contribution at Fig. 1b in Σ. Putting $Z = 1$, we get the equation for the inverse propagator:

$$S^{-1}(p^2, \zeta) = p^2 + m_R^2 + \delta m^2 + \frac{\lambda_R}{16\pi^2} \frac{m_R^2}{(n-4)} \pm \frac{\lambda_R}{2} \mu^{4-n} \Sigma_\pm(\zeta) \tag{59}$$

To find the counterterm $\delta\lambda$ write the sum of vertices in Fig. 2

$$\Gamma_{4)}^{\text{tot}}(p_i, \zeta) = -\lambda_R \mu^{4-n} - \delta\lambda \mu^{4-n} - \frac{3\lambda_R^2 \mu^{4-n}}{16\pi^2(n-4)}$$
$$\pm (\lambda_R \mu^{4-n})^2 [\Omega_\pm(t, \zeta) + \Omega_\pm(u, \zeta) + \Omega_\pm(s, \zeta)] \tag{60}$$

where $s = p_1 + p_2$, $t = p_1 + p_3$, and $u = p_1 + p_4$.

From Eqs. (59) and (60) we get

$$\delta m^2 = -\frac{\lambda_R}{16\pi^2} \frac{m_R^2}{(n-4)} \tag{61}$$

and

$$\delta\lambda = -\frac{3\lambda_R^2}{16\pi^2(n-4)} \tag{62}$$

Assuming S^{-1} $(p^2 = 0, \zeta) = m_\pm(\zeta)$ and $\lambda_\pm(\zeta) = -\Gamma_{4)}^{\text{tot}}$ $(p_i = 0, \zeta)$, we find the equation for the mass parameter and coupling constant for nontrivial topology up to the order $O(\lambda_R^2)$:

$$m_\pm(\zeta) = m_R^2 \pm \frac{\lambda_R}{2} \Sigma_\pm(\zeta) \tag{63}$$

and

$$\lambda_\pm(\zeta) = \lambda_R \mp 3\lambda_R^2\Omega_\pm(\zeta) \tag{64}$$

The equality $\Omega_\pm(p = 0, \zeta) = \Omega_\pm(\zeta)$ results from Eqs. (35) and (52).

To obtain the divergent contributions of the order λ_R^2 to the self-energy, we compute the sum of divergent contributions from Figs. 1c–1g:

$$\begin{aligned}
\Sigma = & -\delta\lambda \frac{m_R^2}{2(16\pi^2)}\left[\frac{2}{(n-4)} + \gamma - 1 + \ln\frac{m_R^2}{4\pi\mu^2}\right] \\
& - \delta m^2 \frac{\lambda_R}{2(16\pi^2)}\left[\frac{2}{(n-4)} + \gamma + \ln\frac{m_R^2}{4\pi\mu^2}\right] \\
& - \frac{1}{(16\pi^2)^2}\frac{m_R^2\lambda_R^2}{(n-4)^2} - \frac{m_R^2\lambda_R^2}{2(16\pi^2)^2}\frac{1}{(n-4)}\left[2\gamma - 1 + 2\ln\frac{m_R^2}{4\pi\mu^2}\right] \\
& - \frac{1}{(16\pi^2)^2}\frac{m_R^2\lambda_R^2}{(n-4)^2} - \frac{m_R^2\lambda_R^2}{(16\pi^2)^2(n-4)}\left[\gamma - 1 + \ln\frac{m_R^2}{4\pi\mu^2}\right] \\
& + \frac{1}{(16\pi^2)^2}\frac{\lambda_R^2}{12(n-4)}(p^2 + 6m_R^2) \\
& \mp \frac{1}{2}\mu^{4-n}\left[\delta\lambda + \frac{3\lambda_R^2}{16\pi^2(n-4)}\right]\Sigma_\pm(\zeta) \\
& \pm \lambda_R\mu^{4-n}\left[\delta m^2 + \frac{\lambda_R m_R^2}{16\pi^2(n-4)}\right]\Omega_\pm(\zeta)
\end{aligned} \tag{65}$$

Equation (65) is simplified for $\delta\lambda$ and δm^2 in the forms (61) and (62). The parameters a_{12} and b_{11} of counterterms (54) and (55) are

$$a_{12} = -3(16\pi^2)^{-1}, \qquad b_{11} = -(16\pi^2)^{-1} \tag{66}$$

In this case divergent terms in Σ do not depend on topology and are written as

$$\Sigma = \frac{2}{(16\pi^2)^2}\frac{\lambda_R m_R^2}{(n-4)^2} + \frac{\lambda_R^2}{(16\pi^2)^2}\frac{1}{12(n-4)}(p^2 + 6m_R^2) \tag{67}$$

The renormalization of the total inverse propagator to the order λ_R^2 may be done with the equation

$$\begin{aligned}
Z[p^2 + m_R^2 + \delta m^2 - \Sigma] = Z\Bigg[&\left(1 - \frac{\lambda_R^2}{(16\pi^2)^2}\frac{1}{12(n-4)}\right)(p^2 + m_R^2) \\
& + \frac{\lambda_R^2 m_R^2}{(n-4)^2}\left(b_{12} - \frac{5}{12}\frac{1}{(16\pi^2)^2}\right)
\end{aligned}$$

$$+ \frac{\lambda_R^2}{(n-4)}\left(b_{22} - \frac{2}{(16\pi^2)^2}\right)\Bigg] \tag{68}$$

It gives

$$b_{12} = \frac{5}{12}\frac{1}{(16\pi^2)^2}, \qquad b_{22} = \frac{2}{(16\pi^2)^2} \tag{69}$$

and the nontrivial field renormalization constant

$$Z = 1 + \frac{\lambda_R^2}{(16\pi^2)^2}\frac{1}{12(n-4)} \tag{70}$$

The above results show that the model is renormalizable in two loops with nontrivial space-time and the renormalizability may be done with topologically independent counterterms, i.e., counterterms of the standard Euclidean model.

The nontrivial contributions of Fig. 1 to the two-point Green's function lead to a ζ dependence of the mass parameter, e.g., the topologically dependent mass-squared to $O(\lambda_R^2)$ is written as

$$m_{\pm}^2(\zeta) = m_R^2\left[1 \pm \frac{\lambda_R}{2} F_{\pm}(m_R\zeta) + O(\lambda_R^2)\right] \tag{71}$$

The Asymptotics of $F_{\pm}(m_R\zeta)$ for $(m_R\zeta \ll 1)$ is given by (A.28). It gives

$$m_{(+)}^2(\zeta) = m_R^2 + \frac{\lambda_R}{24\zeta^2} - \frac{\lambda_R}{8\pi}\frac{m_R}{\zeta} + \ldots \tag{72}$$

for untwisted and

$$m_{(-)}^2(\zeta) = m_R^2 - \frac{\lambda_R}{48\zeta^2} - \frac{\lambda_R}{16\pi^2} m_R^2 \ln(m_R\zeta) + \ldots \tag{73}$$

for twisted fields.

The asymptotics of the coupling constant (64) is written with Eq. (A.24) as

$$\lambda_{(+)}(\zeta) = \lambda_R - \frac{3\lambda_R^2}{16\pi}\left(\frac{1}{m_R\zeta}\right) - \frac{3\lambda_R^2}{16\pi^2}\ln\left(\frac{m_R\zeta}{4\pi}\right) + \ldots \tag{74}$$

for untwisted and

$$\lambda_{(-)}(\zeta) = \lambda_R - \frac{3\lambda_R^2}{16\pi^2}\ln\left(\frac{m_R\zeta}{\pi}\right) + \ldots \tag{75}$$

for twisted fields.

Therefore the nontrivial topology influences the behavior of the coupling constant and the mass parameter.

3. ASYMPTOTIC PROPERTIES OF THE MODEL

The renormalization procedure for the $\lambda\phi^4$ model with nontrivial topology is similar to the renormalization procedure for the finite-temperature model [25, 31]. In case of nontrivial topology the renormalized n-point vertex function $\Gamma_R^{(n)}$ is connected with unrenormalized one $\Gamma^{(n)}$ through the relation

$$\Gamma_R^{(n)}(p, \zeta; \lambda_R, m_R, \mu, \zeta_0) \\ = Z^{n/2}(\lambda_B, m_B, \Lambda, \mu, \zeta_0)\Gamma^{(n)}(p, \zeta; \lambda_B, m_B, \Lambda) \tag{76}$$

where Z is the wavefunction renormalization factor, and λ_R and m_R are the renormalized coupling constant and mass parameter at the renormalization point (μ, ζ_0). If the renormalization point changes, then the physical parameters of the model also change because the physical theory does not depend on the renormalization point at which we renormalize the theory.

The equations which demonstrate how the renormalized vertex functions change when we change the renormalization point can be found from (76) and written as

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda_R)\frac{\partial}{\partial\lambda_R} + m_R\gamma_m\frac{\partial}{\partial m_R} - n\gamma\right]\Gamma_R^{(n)}(p, \zeta; \lambda_R, m_R, \mu, \zeta_0) = 0 \tag{77}$$

and

$$\left[\zeta_0\frac{\partial}{\partial\zeta_0} + \overline{\beta}(\lambda_R)\frac{\partial}{\partial\lambda_R} + m_R\overline{\gamma}_m\frac{\partial}{\partial m_R} - n\overline{\gamma}\right]\Gamma_R^{(n)}(p, \zeta; \lambda_R, m_R, \mu, \zeta_0) = 0 \tag{78}$$

The coefficients in these RG equations are defined by

$$\beta = \mu\frac{\partial\lambda_R}{\partial\mu}, \qquad m_R\gamma_m = \mu\frac{\partial m_R}{\partial\mu} \\ \gamma = \mu\frac{\partial}{\partial\mu}\ln\sqrt{Z} \tag{79}$$

and

$$\overline{\beta} = \zeta_0\frac{\partial\lambda_R}{\partial\zeta_0}, \qquad m_R\overline{\gamma}_m = \zeta_0\frac{\partial m_R}{\partial\zeta_0} \\ \overline{\gamma} = \zeta_0\frac{\partial}{\partial\zeta_0}\ln\sqrt{Z} \tag{80}$$

Now one can use the results obtained in Section 2 to consider the behavior of the running coupling constant with respect to the variation of the topological parameter ζ. The β-functions for different topologies may be found in the asymptotic regime ($m_R\zeta \ll 1$) from (74), (75), and (80) in the following form:

$$\overline{\beta}_{(\pm)}(\lambda_R) = \begin{cases} (3\lambda_R^2/16\pi^2) & (1 - \pi e^t) \\ 3\lambda_R^2/16\pi^2 \end{cases} \tag{81}$$

where the parameter t is given by $t = -\ln(m_R\zeta)$.

The solutions of the differential equation

$$\frac{d\lambda_R(t)}{dt} = \overline{\beta}_{(\pm)}(\lambda_R) \tag{82}$$

with $\overline{\beta}_{(\pm)}$ functions (81) lead to the running coupling constant for untwisted and twisted fields, respectively:

$$\lambda_R^{(+)}(\zeta) = \frac{\lambda_R}{1 + (3\lambda_R/16\pi^2)[(\pi/m_R)(\zeta^{-1} - \zeta_0^{-1}) + \ln(\zeta/\zeta_0)]} \tag{83}$$

and

$$\lambda_R^{(-)}(\zeta) = \frac{\lambda_R}{1 + (3\lambda_R/16\pi^2)\ \ln(\zeta/\zeta_0)} \tag{84}$$

This result predicts the same form of topological behavior of the coupling constant as Eqs. (74) and (75).

4. CONCLUSION AND COMMENTS

We have considered the scalar self-interacting $\lambda\phi^4$ model with nontrivial space-time in the perturbative regime. The space-time structure of this model was proposed as geometrically flat with topology $S^1 \times R^3$. Space-time topology in this case admits twisted and untwisted configurations for self-interacting fields with different properties of the physical parameters for each field configuration. The model is renormalizable in two loops and, as follows from Eq. (72), the effects of nontrival topology lead to the increase of the topology-dependent mass with the decrease of the topological parameter for untwisted field, or to the generation of the mass $m^2(\zeta) \approx \lambda_R(24\zeta^2)^{-1}$ for this field configuration, if the initial field is massless on the tree level. Equation (73) predicts the decrease of the topology-dependent mass of the twisted field with the decrease of the topological parameter and the elimination of the mass for the value of the critical parameter $\zeta_{cr} \approx (\lambda_R/48m_R^2)^{1/2}$. As follows from (74) and (75), the asymptotic behavior of the coupling constant is different for untwisted and twisted fields. For the untwisted field the coupling constant decreases and for the twisted one it increases if the topological parameter ζ of space-time decreases. The effects of nontrivial topology become stronger with the decrease of the topological parameter. It leads to an essential variation of physical parameters in comparison with the results for this model in space-time with trivial topology. In the limit $\zeta^{-1} \to 0$ the coupling constant will obey only Eq. (77), and the asymptotic properties of the coupling constant will be connected only with momentum scaling.

The temperature dependence of the mass parameter and of the coupling constant for a thermal scalar field at the high-temperature limit $m_R/T \ll 1$ coincides with the untwisted field results (72) and (74) for $\zeta^{-1} = T$:

$$m^2(T) = m_R^2 + \frac{\lambda_R}{24} T^2 - \frac{\lambda_R}{8\pi} (m_R T) + \dots \tag{85}$$

and

$$\lambda(T) = \lambda_R - \frac{3\lambda_R^2}{16\pi} \left(\frac{T}{m_R} \right) - \frac{3\lambda_R^2}{16\pi^2} \ln\left(\frac{m_R}{4\pi T} \right) + \dots \tag{86}$$

The technique used in this work seems more convenient then the sum computation in Feynman integrals for this model in the perturbative regime [12, 19].

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APPENDIX

In this section we find the asymptotics of the topological contributions in the mass parameter and coupling constant.

1. The topologically nontrivial contribution $\Sigma_\pm(\zeta)$to the self-energy is

$$\begin{aligned}\Sigma_\pm(\zeta) &= \int \frac{d^4k}{(2\pi)^4} \Delta_\pm(k, \zeta) \\ &= \int \frac{d^4k}{(2\pi)^4} \frac{2\pi i \delta(k^2 + m_R^2)}{\exp(\zeta|k^1|) \mp 1}\end{aligned} \tag{A.1}$$

Let us introduce the new variable $x = ik^1$, assume $\sigma^2 = \mathbf{k}^2 + m_R^2$, and write the δ-function as

$$\delta(k^2 + m_R^2) = \delta(x^2 - \sigma^2) \tag{A.2}$$

Then

$$\Sigma_{\pm}(\zeta) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int dx \, \frac{\delta(x^2 - \sigma^2)}{\exp(\zeta|x|) \mp 1} \tag{A.3}$$

Using the equation

$$\delta(x^2 - \sigma^2) = \frac{1}{2\sigma}[\delta(x - \sigma) + \delta(x + \sigma)] \tag{A.4}$$

we get the integral over the x in (A.3) in the form

$$\begin{aligned} &\int dx \, \frac{1}{\exp(\zeta|x|) \mp 1} \delta(x^2 - \sigma^2) \\ &= \frac{1}{2\sigma} \int dx \, \frac{1}{\exp(\zeta|x|) \mp 1} [\delta(x - \sigma) + \delta(x + \sigma)] \\ &= \frac{1}{\sigma(\exp(\zeta\sigma) \mp 1)} \end{aligned} \tag{A.5}$$

Then $\Sigma_{\pm}(\zeta)$ will be

$$\Sigma_{\pm}(\zeta) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{(\sqrt{\mathbf{k}^2 + m_R^2})[\exp(\zeta\sqrt{\mathbf{k}^2 + m_R^2}) \mp 1]} \tag{A.6}$$

The final form for this contribution is written as

$$\Sigma_{\pm}(\zeta) = \frac{1}{2\pi^2\zeta^2} \int_0^{\infty} \frac{z^2 \, dz}{\sqrt{z^2 + (\zeta m_R)^2}\{\exp[\sqrt{z^2 + (\zeta m_R)^2}] \mp 1\}} \tag{A.7}$$

2. The Function $\Omega_{\pm}(\zeta)$ has the following form:

$$\Omega_{\pm}(\zeta) = \int \frac{d^4 k}{(2\pi)^4} \Delta_0(k)\Delta_{\pm}(k, \zeta) \tag{A.8}$$

Let us introduce the variable $x = ik^1$ and rewrite (A.8) as

$$\Omega_{\pm}(\zeta) = -\int \frac{d^3\mathbf{k}}{(2\pi)^3} \int dx \, \frac{1}{(x^2 - \sigma^2)[\exp(\zeta|x|) \mp 1]} \delta(x^2 - \sigma^2) \tag{A.9}$$

With the help of (A.4) the integral over x is written as

$$\begin{aligned} &\int dx \, \frac{\delta(x^2 - \sigma^2)}{(x^2 - \sigma^2)[\exp(\zeta|x|) \mp 1]} \\ &= \frac{1}{2\sigma} \int dx \, \frac{1}{(x^2 - \sigma^2)[\exp(\zeta|x|) \mp 1]} [\delta(x - \sigma) + \delta(x + \sigma)] \end{aligned} \tag{A.10}$$

We can compute the first integral of (A.10)

$$\frac{1}{2\sigma}\int dx\,\frac{1}{(x^2-\sigma^2)[\exp(\zeta|x|)\mp 1]}\delta(x-\sigma) \tag{A.11}$$

using the expansions

$$\frac{1}{(x-\sigma)(x+\sigma)}=\frac{1}{\Delta x}\left[\frac{1}{2\sigma}-\frac{1}{2\sigma^2}\Delta x+\cdots\right] \tag{A.12}$$

and

$$\begin{aligned}\frac{1}{\exp(\zeta|x|)\mp 1}&=\frac{1}{\exp(\zeta|\sigma+\Delta x|)\mp 1}\\&=\frac{1}{\exp(\zeta\sigma)\mp 1}-\frac{\zeta\exp(\zeta\sigma)}{[\exp(\zeta\sigma)\mp 1]^2}\Delta x+\cdots\end{aligned} \tag{A.13}$$

where $\Delta x = x-\sigma$. Now it is possible to find the principal part of (A.11),

$$\begin{aligned}&\frac{1}{2\sigma}\int dx\,\frac{P}{(x^2-\sigma^2)[\exp(\zeta|x|)\mp 1]}\delta(x-\sigma)\\&\quad=\frac{1}{4\sigma}\int dx\,\frac{P}{\Delta x}\left[\frac{1}{\sigma}-\frac{\Delta x}{\sigma^2}+\cdots\right]\\&\qquad\times\left[\frac{1}{\exp(\zeta\sigma)\mp 1}-\frac{\zeta\exp(\zeta\sigma)}{[\exp(\zeta\sigma)\mp 1]^2}\Delta x+\cdots\right]\delta(\Delta x)\\&\quad=-\frac{1}{4}\int dx\,\frac{P}{\Delta x}\,\delta(\Delta x)\Delta x\left[\frac{1}{\sigma^3[\exp(\zeta\sigma)\mp 1]}+\frac{\zeta\exp(\zeta\sigma)}{\sigma^2[\exp(\zeta\sigma)\mp 1]^2}\right]\\&\quad=-\frac{1}{8}\left[\frac{1}{\sigma^3[\exp(\zeta\sigma)\mp 1]}+\frac{\zeta\exp(\zeta\sigma)}{\sigma^2[\exp(\zeta\sigma)\mp 1]^2}\right]\end{aligned} \tag{A.14}$$

where the principal part of the integral over the variable x is

$$\int dx\,\frac{P}{\Delta x}\,\delta(\Delta x)\Delta x=\frac{1}{2} \tag{A.15}$$

The result of the integration is written as

$$\begin{aligned}&\int dx\left(\frac{P}{x^2-\sigma^2}\right)\frac{1}{\exp(\zeta|x|)\mp 1}\delta(x^2-\sigma^2)\\&\quad=-\frac{1}{4\sigma^3[\exp(\zeta\sigma)\mp 1]}-\frac{\zeta\exp(\zeta\sigma)}{4\sigma^2[\exp(\zeta\sigma)\mp 1]^2}\end{aligned} \tag{A.16}$$

The second term in (A.10) gives the same contribution due to the symmetry ($x \to -x$). Equation (A.9) becomes

$$\Omega_\pm(\zeta) = \frac{1}{4}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\frac{1}{\sigma^3[\exp(\zeta\sigma) \mp 1]} + \frac{\zeta\exp(\zeta\sigma)}{\sigma^2[\exp(\zeta\sigma) \mp 1]^2}\right] \tag{A.17}$$

Integrating (A.17) by parts, we get $\Omega_\pm(\zeta)$ in the form

$$\Omega_\pm(\zeta) = \frac{1}{8\pi^2}\int_0^\infty \frac{dz}{\sqrt{z^2 + (m_R\zeta)^2}[\exp\sqrt{z^2 + (m_R\zeta)^2} \mp 1]} \tag{A.18}$$

3. To find the asymptotics of (A.7) and (A.18), consider the integrals

$$I_\pm(a) = \int_0^\infty \frac{dz}{\sqrt{z^2 + a^2}(\exp\sqrt{z^2 + a^2} \mp 1)} \tag{A.19}$$

and

$$N_\pm(a) = \int_0^\infty \frac{z^2\, dz}{\sqrt{z^2 + a^2}(\exp\sqrt{z^2 + a^2} \mp 1)} \tag{A.20}$$

The expansions ($m\zeta \ll 1$) for $I_\pm(\zeta)$ are found with help of the equations

$$\frac{1}{\exp(u) - 1} = \frac{1}{u} - \frac{1}{2} + 2\sum_{n=1}^\infty \frac{u}{u^2 + 4\pi^2 n^2} \tag{A.21}$$

and

$$\frac{1}{\exp(u) + 1} = \frac{1}{2} - \sum_{n=1}^\infty \frac{u}{u^2 + \pi^2(2n+1)^2} \tag{A.22}$$

Multiplying the integrand of (A.19) by $x^{-\epsilon}$, using the expressions (A.21) and (A.22), and integrating term by term letting $\epsilon \to 0$ at the end, we obtain the asymptotics of $\Omega_\pm(\zeta)$ in the form

$$I_\pm(\zeta) = \begin{cases} \dfrac{\pi}{2a} + \dfrac{1}{2}\ln\left(\dfrac{a}{4\pi}\right) + \dfrac{\gamma}{2} + \cdots \\ -\dfrac{1}{2}\ln\left(\dfrac{a}{\pi}\right) - \dfrac{\gamma}{2} + \cdots \end{cases} \tag{A.23}$$

Then the expansion for $\Omega_\pm(\zeta)$ is

$$\Omega_{\pm}(\zeta) = \begin{cases} \frac{1}{16\pi}\left(\frac{1}{m_R\zeta}\right) + \frac{1}{16\pi^2}\ln\left(\frac{m_R\zeta}{4\pi}\right) + \cdots \\ -\frac{1}{16\pi^2}\ln\left(\frac{m_R\zeta}{\pi}\right) + \cdots \end{cases} \tag{A.24}$$

The contribution $N_{\pm}(a)$ is found from

$$\frac{\partial}{\partial a^2} N_{\pm}(a) = \frac{1}{2} I_{\pm}(a) \tag{A.25}$$

Integrating (A.25) over the variable a with expansion (A.23), we get

$$N_{\pm}(\zeta) = \begin{cases} -\frac{\pi}{2} a - \frac{a^2}{4}\ln a + \frac{a^2}{8}(1 - 2\gamma + 2\ln 4\pi) + \frac{\pi^2}{6} + \cdots \\ \frac{a^2}{4}\ln a - \frac{a^2}{8}(1 - 2\gamma + 2\ln\pi) + \frac{\pi^2}{12} + \cdots \end{cases} \tag{A.26}$$

As a result we can write the equation for $\Sigma(\zeta)$ in the form

$$\Sigma_{\pm}(\zeta) = m_R^2 F_{\pm}(m_R\zeta) \tag{A.27}$$

where the function $F_{\pm}$ has the following expansion:

$$F_{\pm}(a) = \begin{cases} -\frac{1}{4\pi a} - \frac{1}{8\pi^2}\ln a + \frac{1}{16\pi^2}(1 - 2\gamma + 2\ln 4\pi) + \frac{1}{12a^2} + \cdots \\ \frac{1}{8\pi^2}\ln a - \frac{1}{16\pi^2}(1 - 2\gamma + 2\ln\pi) + \frac{1}{24a^2} + \cdots \end{cases} \tag{A.28}$$

REFERENCES

1. B. S. DeWitt, *Dynamical Theory of Group and Fields* (Gordon and Breach, New York, 1965).
2. N. D. Birrell and P. S. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
3. S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge University Press, Cambridge, England, 1989).
4. J. S. Dowker and R. Critchley, *J. Phys. A* **9**, 535–540 (1976).
5. L. H. Ford, *Proc. R. Soc. Lond. A* **368**, 305–310 (1979).
6. B. S. Kay, *Phys. Rev. D* **20**, 3052–3062 (1979).
7. L. S. Brown and G. J. Maclay, *Phys. Rev.* **184**, 1272–1279 (1969).
8. S. Greenspoon and R. Pathria, *Phys. Rev. A* **11**, 1080 (1975).
9. L. H. Ford, *Phys. Rev. D* **21**, 949–957 (1980).
10. L. H. Ford, *Phys. Rev. D* **22**, 3003–3011 (1980).

11. L. H. Ford and T. Yoshimura, *Phys. Lett.* **70A**, 89–91 (1979).
12. D. J. Toms, *Phys. Rev. D* **21**, 928–932 (1980).
13. D. J. Toms, *Phys. Rev. D* **21**, 2805–2817 (1980).
14. L. H. Ford, *Phys. Rev. D* **21**, 933–948 (1980).
15. H. W. Braden, *Phys. Rev. D* **25**, 1028–1038 (1982).
16. I. Kulikov and P. Pronin, *Ann. Phys.* (Leipzig) **7**, 100–102 (1987).
17. I. Kulikov and P. Pronin, *Acta Phys. Pol. B* **20**, 713–722 (1989).
18. P. D. Morley, *Phys. Rev. D* **17**, 598–610 (1978).
19. M. B. Kislinger and P. D. Morley, *Phys. Rev. D* **13**, 2771–2777 (1976).
20. E. Manesis and S. Sakakibara, *Phys. Lett.* **157B**, 287–291 (1985).
21. K. Babu Joseph, V. C. Kuriakose, and M. Sabir, *Phys. Lett.* **115B**, 120–124 (1982).
22. G. Peressutti and B. S. Skagerstam, *Phys. Lett.* **110B**, 406–410 (1982).
23. M. B. Kislinger and P. D. Morley, *Phys. Rev. D* **13**, 2765–2770 (1976).
24. J. C. Collins, *Phys. Rev. D* **10**, 1213–1218 (1974).
25. H. Matsumoto, Y. Nakano, and H. Umezawa, *Phys. Rev. D* **29**, 1116–1124 (1984).
26. L. Dolan and R. Jackiw, *Phys. Rev. D* **9**, 3320–3341 (1974).
27. J. Kapusta, *Nucl. Phys. B* **148**, 461–492 (1979).
28. C. W. Bernard, *Phys. Rev. D* **9**, 3312–3320 (1974).
29. S. Weinberg, *Phys. Rev. D* **9**, 3357–3378 (1974).
30. H. Matsumoto, I. Ojima, and H. Umezawa, *Ann. Phys.* (N.Y.) **152**, 348–375 (1984).
31. Y. Fujimoto and H. Yamada, *Phys. Lett.* **195B**, 231–234 (1987).
32. G. ’t Hooft, *Nucl. Phys. B* **61**, 455 (1973).